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# ASYMPTOTIC SOLUTION OF BOUNDARY VALUE PROBLEMS FOR WEAKLY PERTURBED WAVE EQUATIONS*

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A method that is asymptotic with respect to the small parameter $\varepsilon$ is proposed for solving boundary value problems for weakly linear equations with partial derivatives. Linear travelling waves, defined when $x \geqslant 0$, $t \geqslant 0$ and which only interact on the boundary $x = 0$, are the solution of the unperturbed problems. An asymptotic solution which is uniformly suitable for $t, x = O(\varepsilon^{-1})$ is constructed for the perturbed problem using the method of averaging along the characteristics. A model problem of gas dynamics is considered - the problem of the motion of a piston in a semi-infinite tube.

The problems considered in this paper are usually solved by the method of regular expansion with respect to the parameter $\varepsilon$ /1, 2/. However this method leads to secular terms appearing in the asymptotic solution, which makes the latter unsuitable for values of the arguments $t, x = O(\varepsilon^{-1})$. But large values of t and $x$ are more interesting when analysing weakly linear waves, since the non-linear, dissipative and other factors, which are usually disregarded in the simplest linear models, begin to develop. Asympotic methods enabling us to solve Cauchy's problem for the equations considered below were proposed in /3-6/. Boundary value problems of the "resonator" type were solved in /7, 8/. The technique proposed previously is modified below for problems in which $x \geqslant 0$.

1. In practice, problems in which two travelling waves weakly interact are the ones most frequently analysed. Suppose the behaviour of these waves is described by problem

$$r_t + r_x = \varepsilon f_1[r, s, \varepsilon], \quad s_t - s_x = \varepsilon f_2[r, s, \varepsilon], \quad t > 0, \quad x > 0 \tag{1.1}$$

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$$r(0, x) = g_1(x, \varepsilon),\ s(0, x) = g_2(x, \varepsilon),\ x \geqslant 0 \tag{1.2}$$

$$r(t, 0) + as(t, 0) = h(t, \varepsilon),\ \ t > 0 \tag{1.3}$$

Here $f_1, f_2$ are non-linear operators (algebraic, differential, integral, ...), that are regular with respect to $\varepsilon$ in the neighbourhood of the point $\varepsilon = 0$. The functions $g_1, g_2, h$ are periodic with the period $\Lambda$ and are also regular with respect to $\varepsilon$. Both the initial data and the solution of the problem are assumed to be fairly smooth. In (1.1) and henceforth the letters $t, x, \tau, \xi, y, z$, used as the lower indices, indicate the partial derivatives with respect to these variables.

When $\varepsilon = 0$ the solution of Cauchy's problem (1.1), (1.2) is uniquely defined in the domain $D_1 = \{x \geqslant t > 0\}$. In the domain $D_2 = \{t > x > 0\}$ the solution of problem (1.1), (1.3) which satisfies the condition of continuity of the function $s(t, x)$ on the line $x = t$, is also uniquely defined. The differential properties of the solution of the linear problem depend on the properties of the functions $g_1, g_2, h$. In particular, the smoothness of the function $r(t, x)$ on the line $x = t$ follows from the matching conditions of the derivatives of the functions $g_1, g_2, h$ at the point O /9/.

Similar considerations are used to construct an asymptotic solution when $\varepsilon \neq 0$. Suppose the curve $x = X_1(t) = t + x_1(\varepsilon t)$, where $x_1(0) = 0$ divides the domain $D = \{t > 0,\ x > 0\}$ into two subdomains $D_1(\varepsilon)$ and $D_2(\varepsilon)$, which when $\varepsilon = 0$ agree with the domains $D_1, D_2$ defined above. As will be clear from what follows, this division depends on the principal terms of the asymptotic solution, which is sought in the form

$$\begin{gathered} r \sim v_{10}(\tau, y) + \sum_{k\geqslant 1} \varepsilon^k [v_{1k}(\tau, y) + r_{1k}(\tau, y, z)] \\ s \sim w_{10}(\tau, z) + \sum_{k\geqslant 1} \varepsilon^k [w_{1k}(\tau, z) + s_{1k}(\tau, y, z)] \text{ in } D_1(\varepsilon) \end{gathered} \tag{1.4}$$

$$\begin{gathered} r \sim v_{20}(\tau, \xi, y) + \sum_{k\geqslant 1} \varepsilon^k [v_{2k}(\tau, \xi, y) + r_{2k}(\tau, \xi, y, z)] \\ s \sim w_{20}(\tau, \xi, z) + \sum_{k\geqslant 1} \varepsilon^k [w_{2k}(\tau, \xi, z) + s_{2k}(\tau, \xi, y, z)] \ \text{ in } D_2(\varepsilon) \\ (\tau = \varepsilon t,\ \xi = \varepsilon x,\ \ y = x - t,\ \ z = x + t) \end{gathered} \tag{1.5}$$

The substitution of (1.4) into (1.1) and the expansion of the operators $f_1, f_2$ in powers of $\varepsilon$ reduces to the recurrent systems

$$\begin{gathered} 2r_{1,\,k+1,\,z} = f_{1k}[v_{10}, w_{10}, \ldots, v_{1k}, w_{1k}, r_{1k}, s_{1k}] - v_{1k\tau} - r_{1k\tau} \\ -2s_{1,\,k+1,\,y} = f_{2k}[v_{10}, w_{10}, \ldots, v_{1k}, w_{1k}, r_{1k}, s_{1k}] - w_{1k\tau} - s_{1k\tau},\ \ k \geqslant 0 \end{gathered} \tag{1.6}$$

Here $f_{ik}$ are coefficients for $\varepsilon^k$, obtained after expanding the operators $f_i$, $i = 1, 2$. If we assume that the functions $v_{10}, w_{10}, \ldots, v_{1k}, w_{1k}, r_{1k}, s_{1k}$ are already defined and periodic with respect to $y$ and $z$ with the period $\Lambda$, the conditions for the absence of secular terms in $r_{1,\,k+1}, s_{1,\,k+1}$ (and their periodicity with respect to $y$ and $z$) take the form

$$\begin{gathered} v_{1k\tau} = \langle f_{1k} - r_{1k\tau}\rangle_1,\ \ w_{1k\tau} = \langle f_{2k} - s_{1k\tau}\rangle_2,\ \ k \geqslant 0 \\ \langle F(y, z)\rangle_1 = \frac{1}{\Lambda}\int_0^\Lambda F(y, z)\,dz,\ \langle F(y, z)\rangle_2 = \frac{1}{\Lambda}\int_0^\Lambda F(y, z)\,dy \end{gathered} \tag{1.7}$$

Eqs.(1.6) and (1.7) are supplemented by the initial conditions obtained after subsituting (1.4) into (1.2)

$$v_{10}(0, y) = g_1(y, 0),\ y \geqslant 0;\ w_{10}(0, z) = g_2(z, 0),\ z \geqslant 0 \tag{1.8}$$

$$\begin{gathered} v_{1k}(0, y) = w_{1k}(0, z) \equiv 0,\ \ r_{1k}(0, y, y) = g_{1k}(y), \\ s_{1k}(0, z, z) = g_{2k}(z),\ \ k \geqslant 1 \end{gathered} \tag{1.9}$$

($g_{1k}, g_{2k}$ are the coefficients of the expansions with respect to $\varepsilon$ of the functions $g_1, g_2$). The arbitrariness in choosing the initial conditions (1.9) does not affect the definition of the functions $v_{1k} + r_{1k}, w_{1k} + s_{1k}$, since when $k \geqslant 1$ Eqs.(1.7) are linear. The linearity of these functions also reduces to the fact that their solutions are defined when the principal term of the asymptotic solution is defined, i.e. $v_{10}, w_{10}$.

When $k = 0$ the first equation of (1.7) has the form

$$v_{10\tau} = \langle f_1[v_{10}, w_{10}, 0]\rangle_1 \tag{1.10}$$

Two cases are possible when defining the domain $D_1(\varepsilon)$.

1°. Suppose the function $v_{10}$ is defined when $\tau \geqslant 0,\ y \geqslant y_1(\tau)$, where $y_1(\tau)$ is a fairly smooth function and $y_1(0) = 0$. Then $x_1(\varepsilon t) \equiv y_1(\varepsilon t)$. This case occurs if, for example, $f_1$ is a non-linear function of $r$ and $s$. We can then consider (1.10) as an ordinary differential equation with respect to $\tau$ and $y_1(\tau) \equiv 0$. We can also define the function $y_1(\tau)$ in the case when $f_1$ is a first-order quasilinear differential operator.

$2^{\circ}$. The domain of definition of the function $v_{10}$ remains unknown, since there are not enough conditions (1.8) to define it unambiguously. This situation occurs in the parabolic case. For example, when $f_1 = r_{xx}$ Eq.(1.10) takes the form $v_{10\tau} = v_{10yy}$. However, we can then also assume that the curve $y = y_1(\tau)$, which is a free boundary for the domain of definition of the function $v_{10}$, exists.

2. The expansion (1.5) is constructed in a similar way, with the sole difference that the following recurrence equations are obtained instead of (1.6), (1.7):

$$2r_{2,k+1,z} = f_{1k}[v_{20}, w_{20}, \ldots, r_{2k}, s_{2k}] - v_{2k\tau} - v_{2k\xi} - r_{2k\tau} - r_{2k\xi} \tag{2.1}$$
$$-2s_{2,k+1,y} = f_{2k}[v_{20}, w_{20}, \ldots, r_{2k}, s_{2k}] - w_{2k\tau} + w_{2k\xi} - s_{2k\tau} + s_{2k\xi}$$

$$v_{2k\tau} + v_{2k\xi} = \langle f_{1k} - r_{2k\tau} - r_{2k\xi}\rangle_1 \tag{2.2}$$
$$w_{2k\tau} - w_{2k\xi} = \langle f_{2k} - s_{2k\tau} + s_{2k\xi}\rangle_2, \quad k \geqslant 0$$

Substitition of (1.5) into (1.3) leads to the relations

$$v_{20}(\tau, 0, -t) + aw_{20}(\tau, 0, t) = h(t, 0), \quad t \geqslant 0 \tag{2.3}$$

$$v_{2k}(\tau, 0, -t) + aw_{2k}(\tau, 0, t) = 0 \tag{2.4}$$
$$r_{2k}(\tau, 0, -t, t) + as_{2k}(\tau, 0, -t, t) = h_k(t), \quad t > 0, \quad k \geqslant 1$$

There are not enough conditions (2.3), (2.4) to define the required functions uniquely; the continuity of $s(t, x)$ on the curve $x = X_1(t)$ is therefore required in addition (by analogy with the linear case). The equation of this curve can be rewritten in the slow variables $\xi = \tau + \varepsilon x_1(\tau)$, and the conditions of continuity take the form

$$w_{2k}(\tau, \tau + \varepsilon x_1(\tau), z) = w_{1k}(\tau, z) \tag{2.5}$$
$$s_{2k}(\tau, \tau + \varepsilon x_1(\tau), x_1(\tau), z) = s_{1k}(\tau, x_1(\tau), z)$$

The conditions (2.5) are inconvenient for solution in practice, since they explicitly contain $\varepsilon$. If the functions $w_{2k}, s_{2k}$ are fairly smooth with respect to $\xi$, we can expand them in a Taylor series, and can re-expand conditions (2.5) in powers of $\varepsilon$. The conditions $w_{2k}(\tau, \tau, z) = w_{1k}(\tau, z)$, $k \geqslant 0$, $s_{21}(\tau, \tau, x_1(\tau), z) = s_{11}(\tau, x_1(\tau), z) - w_{20\xi}(\tau, \tau, z)\, x_1(\tau)$ etc., will be obtained.

3. When justifying the above method it is natural to assume that the solution of the initial problem exists in the large domain $0 \leqslant t \leqslant t_0\varepsilon^{-1}$, $0 \leqslant x \leqslant x_0\varepsilon^{-1}$ and the linearized problem (1.1)-(1.3) is stable with respect to the perturbations of the right-hand sides in (1.1). In this case the closeness of the exact and asymptotic solutions is proved using the standard method if the solvability of problems (1.6)-(1.9) and (2.1)-(2.5) is proved beforehand. A difference between case $1^{\circ}$ and $2^{\circ}$ emerges here.

In case $1^{\circ}$ problems (1.6)-(1.9) and (2.1)-(2.5) are solved sequentially and the solvability can be proved separately for the initial and boundary value problems. However the terms of the expansions (1.4) and (1.5) constructed in this way, or their derivatives, may be discontinuous on the curve $x = X_1(t)$, which hinders the estimate of the closeness of the exact and asymptotic solutions in the neighbourhood of this curve. In the simplest case, when $f_1, f_2$ are non-linear functions and $y_1(\tau) \equiv 0$, the curve $x = X_1(t) \equiv t$ is defined exactly and a strict justification of the asymptotic method is possible on the basis of /9/.

In case $2^{\circ}$ problems (1.6)-(1.9) and (2.1)-(2.5) for each number $k$ should be considered simultaneously, assuming the curve $x = X_1(t)$ (or $y = y_1(\tau)$) is indeterminate and imposing additional requirements of smoothness on this curve. The solvability of the problem obtained (an analogue of Stefan's problem for the heat conduction equation) is non-trivial.

4. Although the case of the interaction of two waves is much simpler, we can also generalize the above technique for $n \geqslant 3$ waves. Suppose the behaviour of these waves is described by the problem

$$u_{it} + \lambda_i u_{ix} = \varepsilon f_i[u, \varepsilon], \quad i = 1, \ldots, n, \quad t > 0, \quad x > 0 \tag{4.1}$$

$$u_i(0, x) = g_i(x, \varepsilon), \quad x \geqslant 0, \quad i = 1, \ldots, n \tag{4.2}$$

$$u_j(t, 0) + \sum_{i=m+1}^{n} a_{ji} u_i(t, 0) = h_j(t, \varepsilon), \quad t > 0, \quad j = 1, \ldots, m \tag{4.3}$$

The numbers $\lambda_i$ are numbered in decreasing order and the inequality $\lambda_m > 0 > \lambda_{m+1}$, $1 \leqslant m < n$ holds for some $m$; $f_i$ are non-linear operators that are active in $u = (u_1, \ldots, u_n)$, such that the solution of problem (4.1)-(4.3) is periodic with respect to $x$ with period $\Lambda$. For this periodicity, as follows from the above case $\varepsilon = 0$, we need the periodicity of the functions $g_i$ with period $\Lambda$, the periodicity of the functions $h_j$ with period $\lambda_j\Lambda$ respectively and the existence of the integral $n_{ij}$, such that

$$\lambda_j = n_{ij}\lambda_i, \quad \forall j = m+1, \ldots, n, \quad \forall i = 1, \ldots, m \tag{4.4}$$

In addition, to eliminate small denominators in the asymptotic solution we need to require (as in /3/) the existence of $c \neq 0$ and of the integral $m_{ij}$, such that

$$c(\lambda_i - \lambda_j) = m_{ij}\Lambda, \quad i, j = 1, \ldots, n \tag{4.5}$$

The constraints imposed guarantee the periodicity with respect to $x$ of all the terms of the asymptotic solution. These constraints can be substantially weakened for non-periodic solutions (para.6).

To construct an asymptotic solution of problem (4.1)-(4.3), it is assumed that the non-intersecting curves $x = X_j(t) \equiv \lambda_j t + x_j(\varepsilon t)$, $j = 1, \ldots, m$ exist, such that $x_j(0) = 0$ and $0 < x_m(t) < \ldots < x_2(t) < x_1(t) < \infty$ when $t > 0$. These curves divide the domain $D$ into $m+1$ subdomains $D_j(\varepsilon)$, $j = 1, \ldots, m+1$ where $D_1(\varepsilon) = \{t > 0, \ x > X_1(t)\}$, $D_j(\varepsilon) = \{t > 0, \ X_j(t) < x < X_{j-1}(t)\}$, $j = 2, \ldots, m$, $D_{m+1}(\varepsilon) = \{t > 0, 0 < x < X_m(t)\}$.

In each subdomain $D_j(\varepsilon)$ the asymptotic solution is sought in the form

$$u_i \sim v_{ij0}(\tau, \xi, y_i) + \sum_{k \geqslant 1} \varepsilon^k [v_{ijk}(\tau, \xi, y_i) + w_{ijk}(\tau, \xi, y_i, t)] \tag{4.6}$$

$$y_i = x - \lambda_i t, \ \tau = \varepsilon t, \ \xi = \varepsilon x$$

where the coefficients of the expansion in $D_1(\varepsilon)$ do not depend on $\xi$. The substitution of (4.6) into (4.1) and the requirement that there are no terms in $w_{ijk}$ which are secular with respect to $t$ lead to equations in $D_j(\varepsilon)$

$$v_{ijk\tau} + \lambda_i v_{ijk\xi} = \langle f_{ik} - w_{ijk\tau} - \lambda_i w_{ijk\xi}\rangle_i, \quad i = 1, \ldots, n \tag{4.7}$$

$$w_{ij,k+1,t} = f_{ik} - w_{ijk\tau} - \lambda_i w_{ijk\xi} - v_{ijk\tau} - \lambda_i v_{ijk\xi}, \quad k \geqslant 0 \tag{4.8}$$

The coefficients of the expansion of the operator $f_i$ in powers of $\varepsilon$ are denoted by $f_{ik}$; $\langle f_{ik}\rangle_i$ is the average of the function $f_{ik}$, calculated along the straight lines $y_i = x - \lambda_i t = \text{const}$. Conditions (4.4) and (4.5) guarantee the periodicity of $f_{ik}$ along the corresponding stright lines; therefore

$$\langle F(t, x)\rangle_i = \frac{1}{c}\int_0^c F(s, x - \lambda_i t + \lambda_i s)\, ds$$

To define the functions $v_{ijk}, w_{ijk}$ uniquely, it is necessary to formulate boundary value problems for (4.7), (4.8). Cauchy's problem is formulated in the domain $D_1(\varepsilon)$, and the initial conditions are specified when $\tau = 0$, $y_i \geqslant 0$ and derived from (4.2). If we have an analogue of case 1°, this problem has a unique solution and the function $x_1(\varepsilon t)$ can be defined using this solution. The values of the functions $v_{i2k}, w_{i2k}$, $i = 2, \ldots, n$, which equal those of the corresponding functions $v_{i1k}$, $w_{i1k}$ when $\xi = \lambda_1\tau + \varepsilon x_1(\tau)$, can be specified on the boundary $x = X_1(t)$ of the domain $D_2(\varepsilon)$. In the same way, the agreement of the values of the functions $v_{i,j+1,k}$ and $v_{ijk}, w_{i,j+1,k}$ and $w_{ijk}$ for $1 \leqslant i \leqslant n, i \neq j$ is required on the curves $x = X_j(t)$ (or $\xi = \lambda_j\tau + \varepsilon x_j(\tau)$), $j = 2, \ldots, m$. Finally, when $x = 0$ (or $\xi = 0$) from (4.3) we obtain $m$ boundary conditions for $v_{i,m+1,k}, w_{i,m+1,k}$, $i = 1, \ldots, m$.

In the case when $f_i$ are non-linear functions, the curves $x = X_j(t)$, $j = 1, \ldots, m$ can be considered known $(x_j(\tau) \equiv 0)$, which considerably simplifies the solvability of the boundary value problem obtained. If $f_i$ are first-order quasilinear differential operators, the curves $x = X_j(t)$, $j = 1, \ldots, m$ must be determined during the solution, i.e. an analogue of case 2° is obtained.

5. The proposed method can be applied to the problem of the motion of a piston in a semi-infinite tube filled with an ideal gas. The boundary value problem is described by the relations

$$R_t + \psi(R - S)R_x = 0, \quad S_t - \psi(R - S)S_x = 0 \tag{5.1}$$

$$R = u + \frac{P_0}{\mu}\left[\left(\frac{p}{P_0}\right)^\mu - 1\right], \quad S = 2u - R$$

$$\psi = \left(\frac{p}{P_0}\right)^\varkappa = \left(1 + \frac{\mu}{P_0}\frac{R - S}{2}\right)^\nu, \quad \mu = \frac{\gamma - 1}{2\gamma},$$

$$\varkappa = \frac{\gamma + 1}{2\gamma}, \quad \nu = \frac{\gamma + 1}{\gamma - 1}$$

Here $R$ and $S$ are Riemann invariants, $u$ is the velocity, $p$ is the pressure, $P_0 = \text{const}$ is the unperturbed value of the pressure, $\gamma > 1$ is the index in the equation of state $p = \text{const}\cdot\rho^\gamma$, and $\rho$ is the specific density of the gas. The dimensionless Lagrangian coordinates $t$ and $x$, are chosen in (5.1), such that $\psi = 1$ when $p = P_0$.

Suppose the following boundary conditions are specified for (5.1):

$$u(0,x)=\varepsilon u_0(x),\ \ p(0,x)=P_0+\varepsilon p_0(x),\ \ x\geqslant 0,\ \ u(t,0)=\varepsilon u_1(t),\ t>0$$

($\varepsilon \ll 1$ is a parameter characterizing the smallness of the perturbations with respect to the state of equilibrium $u=0,\ p=P_0$). This problem was solved in /10/ in the case when there are no initial perturbations ($u_0(x)=p_0(x)\equiv 0$), which led to a simplified version of the averaged equations obtained below for the principal term of the asymptotic solution.

The change of variables $R=\varepsilon r,\ S=\varepsilon s$ leads to the problem (1.1)-(1.3) with

$$a=1,\ h(t)=2u_1(t)$$
$$\varepsilon f_1=[1-\psi(\varepsilon(r-s))]\,r_x,\ \ \varepsilon f_2=[\psi(\varepsilon(r-s))-1]\,s_x$$
$$g_{1,2}=u_0(x)\pm\frac{P_0}{\mu\varepsilon}\left[\left(1+\frac{\varepsilon p_0(x)}{P_0}\right)^{\mu}-1\right]$$
$$\psi(\varepsilon(r-s))=1+\varepsilon\psi_1(r-s)+O(\varepsilon^2),\ \ \psi_1=\frac{\gamma+1}{4P_0\gamma}$$

In the domain $D_1(\varepsilon)$ the principal terms of the asymptotic solution (1.4) satisfy the problems

$$v_{10\tau}+\psi_1(v_{10}-\langle w_{10}\rangle_1)\,v_{10y}=0,\ \ v_{10}(0,y)=u_0(y)+p_0(y) \tag{5.2}$$
$$w_{10\tau}+\psi_1(w_{10}-\langle v_{10}\rangle_2)\,w_{10z}=0,\ w_{10}(0,z)=u_0(z)-p_0(z) \tag{5.3}$$

The integration of Eqs.(5.2) and (5.3) with respect to $y$ and $z$ leads to the equations $\langle w_{10}\rangle_1=\delta_1=\langle u_0\rangle-\langle p_0\rangle,\ \ \langle v_{10}\rangle_2=\delta_2=\langle u_0\rangle+\langle p_0\rangle$, therefore problems (5.2) and (5.3) can be solved independently. Physically, this indicates that the principal terms of the Riemann invariants $r$ and $s$ do not interact in the domain $D_1(\varepsilon)$. This behaviour is similar to the case of linear waves. But whereas linear waves preserve the initial smoothness, discontinuities that appear when $\tau=\tau_0>0$ or $t=\tau_0\varepsilon^{-1}$ are formed in the simple waves $v_{10}, w_{10}$ from as many smooth periodic profiles as desired.

Confining ourselves to smooth flows, we can construct the domain $D_1(\varepsilon)$ like the domain of determinacy of the solution of problem (5.2). The characteristic of Eq.(5.2) $y=y_1(\tau)$ which passes through the point $(0,0)$, has the form $y_1(\tau)=\alpha_1\tau$, where $\alpha_1=\psi_1(v_{10}(0,0)-\delta_1)$.

The following problem is solved in the domain $D_2(\varepsilon)$:

$$v_{20\tau}+v_{20\xi}+\psi_1(v_{20}-\langle w_{20}\rangle_1)\,v_{20y}=0,\ \ \xi>0,\ \ y<0 \tag{5.4}$$
$$w_{20\tau}-w_{20\xi}+\psi_1(w_{20}-\langle v_{20}\rangle_2)\,w_{20z}=0,\ \ 0<\xi<\tau(1+\varepsilon\alpha_1),\ z>0 \tag{5.5}$$
$$w_{20}(\tau,\tau(1+\varepsilon\alpha_1),z)=w_{10}(\tau,z),\ z\geqslant 0 \tag{5.6}$$
$$v_{20}(\tau,0,-y)+w_{20}(\tau,0,y)=h(y),\ y>0 \tag{5.7}$$

The waves $v_{20}$ and $w_{20}$ do not interact either, since the integration of (5.4) with respect to $y$, and (5.5) with respect to $z$ reduces to the equations $\langle w_{20}\rangle_1=\delta_3,\ \langle v_{20}\rangle_2=\delta_4$. It follows from (5.6) and (5.7) that $\delta_3=\delta_1$ and $\delta_3+\delta_4=\langle h(t)\rangle$. Therefore problem (5.5), (5.6) is first solved, and $w_{20}(\tau,0,y)$ is defined, and then problem (5.4), (5.7).

The projection of the pattern of Eq.(5.4), which passes though the point $(0,0,0)$, on to the plane $(\xi,y)$ has the form $y_2(\xi)=\alpha_2\xi$, where $\alpha_2=\psi_1[h(0)-w_{20}(0,0,0)-\delta_3]=\psi_1[2u_1(0)-w_{10}(0,0)-\delta_3]$. Consequently, the function $v_{20}$ is defined when $\tau\geqslant 0,\ \xi\geqslant 0,\ y\leqslant\alpha_2\xi$. However, the domain of determinacy of the function $v_{20}$ which is constructed in this way does not generally agree with $D_2(\varepsilon)=D/D_1(\varepsilon)$. Three cases are possible.

1) $u_0(0)=u_1(0)$ - the initial velocity of the gas at the point $x=0$ agrees with the velocity of motion of a piston at the initial instant $t=0$. Then $\alpha_2=\alpha_1$ and the straight lines $l_1$: $y=y_1(\tau)$, $l_2$: $y=y_2(\xi)$ in the plane $(x,t)$ "almost" agree. In fact, the straight line $l_1$ is described by the equation $x=(1+\varepsilon\alpha_1)t$, and $l_2$ is described by the equation $x=(1-\varepsilon\alpha_1)^{-1}t=(1+\varepsilon\alpha_1)t+\varepsilon^2\alpha_1^2t+\ldots$. Consequently, the distance between $l_1$ and $l_2$ is the quantity $O(\varepsilon)$ when $t=O(\varepsilon^{-1})$. We can also remove this gap, however, if we modify expansion (1.5) slightly.

Suppose

$$\xi=\varepsilon\eta x,\ \eta=1+\varepsilon\eta_1+\varepsilon^2\eta_2+\ldots \tag{5.8}$$

where the constants $\eta_i,\ i\geqslant 1$ are determined from the matching condition of the straight lines $l_1, l_2$. The modification of the slow variable $\xi$ does not alter Eqs.(5.4), (5.5), since the effect of the constants $\eta_i$ develops in equations for $v_{2k}, w_{2k}, r_{2k}, s_{2k}$ when $k\geqslant 1$. Therefore, we can determine the coefficient $\eta$ from the equation $1+\varepsilon\alpha_1=(1-\varepsilon\eta\alpha_1)^{-1}$ and can obtain $\eta=(1+\varepsilon\alpha_1)^{-1}=1-\varepsilon\alpha_1+\varepsilon^2\alpha_1^2-\ldots$ With this determination of $\eta$ the equation $x=(1+\varepsilon\alpha_1)t$ is written in slow variables in the form $\xi=\tau$, which enables us to get rid of the parameter $\varepsilon$ under condition (5.6), which takes the form

$$w_{20}(\tau,\tau,z)=w_{10}(\tau,z),\ z\geqslant 0$$

2) $u_0(0)>u_1(0)$ - an extreme particle of the gas which is situated at the point $x=0$ when $t=0$ breaks away from the piston. Then $\alpha_2<\alpha_1$ and the line $l_1$ in the $(x,t)$ plane

lies "below" the line $l_2$. Therefore, not one of the functions $v_{10}, v_{20}$ is defined in the sector $L$, formed by these lines. The value of the function $v_{10}$ on the line $l_1$ equals $\nu_1 = u_0(0) + p_0(0)$, and the value of the function $v_{20}$ on the line $l_2$ equals $\nu_2 = 2u_1(0) - u_0(0) + p_0(0) < \nu_1$. In the sector $L$ we can supplement the definition of the function $v_{20}$ using the formula

$$v_{20} = \frac{y}{\psi_1 \xi} + \delta_3, \quad \alpha_2 \leqslant \frac{y}{\xi} \leqslant \frac{\alpha_1}{1 + \varepsilon\alpha_1} \tag{5.9}$$

Function (5.9) inside $L$ satisfies Eq.(5.4), and takes the value $\nu_2$ on the line $l_2$, and the following value on the line $l_1$:

$$\frac{\nu_1 + \varepsilon\alpha_1\delta_1}{1 + \varepsilon\alpha_1} = \nu_1 + O(\varepsilon) \tag{5.10}$$

In this case the modification (5.8) does not enable us to prove the gap between the lines $l_1, l_2$, but its use eliminates $\varepsilon$ under condition (5.6) and removes the discrepancy in the boundary condition (5.10). The function (5.9) describes the rarefaction wave that appears at the point $x = 0$, $t = 0$.

3) $u_0(0) < u_1(0)$ - the piston moves faster and compresses the gas. Then $\alpha_2 > \alpha_1$, the line $l_1$ in the $(x, t)$ plane is "higher" then the line $l_2$ and in the sector $L$ the functions $v_{10}, v_{20}$ define different asymptotic representations for the Riemann invariant $r$. This situation is impossible physically; we are obliged therefore to assume that the curve $x = x_s(t)$, when passing through which $v_{10}$ and $v_{20}$ alter stepwise and which belongs to the sector $L$, separates the domains $D_1(\varepsilon)$, $D_2(\varepsilon)$. Suppose $B \equiv B(t) = x_s'(t)$, $u^+(t) = u(t, x_s(t) + 0)$, $u^-(t) = u(t, x_s(t) - 0)$, $[u] = u^+ - u^-$ etc. Then the conditions on the discontinuity for the initial problem have the form

$$B[\rho^{-1}] + [u] = 0, \; B[u] = [p] \tag{5.11}$$

In the case of small perturbations $B = \pm 1 + O(\varepsilon)$. Since the discontinuity from the point $(0, 0)$ can only move to the right,

$$B = 1 + \varepsilon b \tag{5.12}$$

It follows, by substituting (5.12) into (5.11) and expressing $u, p, \rho$ in terms of the Riemann invariants, that the functions $w_{10}, w_{20}$ are continuous in this discontinuity, and $b = \psi_1((v_{10}^+ + v_{20}^-)/2 - w_{10}) + O(\varepsilon)$. The equation for the line of discontinuity can be represented in the form

$$\frac{d(x_s(t) - t)}{dt} = \varepsilon\psi_1\left[\frac{v_{10}^+ + v_{20}^-}{2} - w_{10}(\tau, x_s(t) + t)\right] + O(\varepsilon^2) \tag{5.13}$$

This equation shows that the Riemann invariants interact at the discontinuity (as on the bound $x = 0$) that is already of the zeroth order.

In the simplest case $w_{10} = \delta_1 = \text{const}$ the principal part of Eq.(5.13) can be written in the form

$$\frac{dy_s(\tau)}{d\tau} = \psi_1\left(\frac{v_{10}^+ + v_{20}^-}{2} - \delta_1\right) \tag{5.14}$$

where $y = y_s(\tau)$ is a line of discontinuity for $v_{10}(\tau, y)$. In this case the functions $v_{10}, v_{20}$ can be constructed independently of each other, and consequently the function $y_s(\tau)$ is obtained after solving the ordinary differential Eq.(5.14).

A simpler situation was considered in /10/: $w_{10} = w_{20} = v_{10} = 0$. In general the curve $x = x_1(t)$ can only be obtained after solving problems (5.2)-(5.7), (5.13), which is an analogue of case 2° from Sect.1, whilst an explicit dependence on $\varepsilon$ is not excluded in (5.13).

**6.** The above technique was considered for periodic functions. However averaging along the characteristics was also possible in the general case. Most of the results obtained in Sects.1-5 also hold in the classes of functions $F(t, x)$ for which the following means - which are uniform with respect to t and $x_0$ - exist:

$$\langle F(t, x)\rangle = \lim_{X \to +\infty} \frac{1}{X} \int_{x_0}^{x_0 + X} F(t, x)\, dx$$

These can be conditionally periodic functions, and functions that are constant when $x \geqslant x_*$; or that approach constants fairly rapidly as $x \to +\infty$. Quite interesting results can be obtained in these classes.

For example, if generalized means in the initial and boundary functions exist in the problem from Sect.5, and the following inequalities hold:

$$u_1(0) \leqslant u_0(0), \; |p_0'(x)| \leqslant u_0'(x), \; x \geqslant 0$$

$$2u_1'(t) \geqslant w_{20z}(\tau, 0, t), \; \tau, \; t \geqslant 0$$

the principal term of the asymptotic solution is continuous for all $t \geqslant 0, x \geqslant 0$.

The technique can also be used formally in the case when the functions $f_i$ in (4.1), $g_i$ in (4.2) and $h_k$ in (4.3) depend on the slow variables $\tau, \xi$, or non-linear conditions that are solved with respect to the functions $u_j(t, 0), j = 1, \ldots, m$ are considered instead of the linear boundary conditions (4.3).

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# EQUATIONS DESCRIBING THE PROPAGATION OF NON-LINEAR QUASITRANSVERSE WAVES IN A WEAKLY NON-ISOTROPIC ELASTIC BODY*

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Approximate equations are obtained, describing the propagation of a non-linear quasitransverse wave of low amplitude, or a group of such waves, in a nearly isotropic elastic medium, when the characteristic velocities of the waves (dependent on their polarization) differ from one another by a small quantity.

The equations of non-linear geometrical acoustics, and the short-wave equations, are well-known /1-9/; they are obtained on the basis of the fact that waves connected with one family of characteristic surfaces can be propagated. Disturbances linked with other characteristics interact weakly with these waves, by virtue of the assumptions that the amplitude is small and the waves are quasiplane. It is also important that, due to the small difference in the wave velocities, their interaction time is small, if the length of the groups of waves in question is finite.

With small anisotropy and non-linearity, the equations of the theory of elasticity have two properties: the two characteristic velocities corresponding to quasitransverse waves are close, and the non-linearity is extremely small. In the absence of anisotropy (including that due to initial deformation), the non-linearity appears only in the cubic terms; while if there is small anisotropy, quadratic terms also make an appearance, though with small coefficients. Due to the closeness of the quasi-transverse wave velocities, they interact together long-term, so that the evolution of these waves can be studied by considering two waves simultaneously.

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